# On Modalities in Girard's Transcendental Syntax

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In his transcendental syntax, Girard connects proof-nets with unification theory by introducing several new technical devices, such as stars and constellations. At the same time, transcendental syntax imposes restrictions on the use of modalities and introduces the new logical connectives:

> Transcendental syntax tells us that there can be nothing like the exponentials $!A$ and $?A$; one must thus operate a strategic retreat in direction of intuitionistic implication, in other terms, to restrict the use of $!A$ and $?A$ to combinations $A \varowedge B := \sim(A \Rightarrow \sim B)$ (i.e. $!A \otimes B$) and $A \ltimes B := \sim A \Rightarrow B$ (i.e. $?A \parr B$). Since $!A$ can be defined by means of $!A := A \parr 1$, this means that the multiplicative constants are rejected: they have no conditions of possibility. (Girard 2015:841)[1].

However, the impact of such restrictions on modalities on the proof-theoretic strength of the system has not been discussed. In this short note, we clarify this point.

While Girard develops transcendental syntax within its own conceptual framework, our aim is to examine its proof-theoretic strength from the perspective of existing logical systems. For this purpose, we base our analysis on the sequent calculus presented in Girard (2015), which is reproduced in Figure 1.

[1] Jean-Yves Girard, Transcendental syntax I:deterministic case, Mathematical Structures in Computer Science, vol. 27, no. 6: Computing with Lambda-terms. A Special Issue Dedicated to Corrado Böhm for his 90th Birthday, pp. 827–849, 2017 (first published online in 2015). `https://doi.org/10.1017/S0960129515000407`

$$\frac{}{\vdash A, \sim A}\ (\textsc{Id}) \qquad \frac{\vdash \Gamma, \underline{\Delta}, A \otimes \sim A}{\vdash \Gamma, \underline{\Delta}, [A \otimes \sim A]}\ \textsc{Cut}$$

$$\frac{\vdash \Gamma, \underline{\Delta}, \underline{A \otimes \sim A}}{\vdash \Gamma, \underline{\Delta}, \underline{[A \otimes \sim A]}}\ (\underline{\textsc{Cut}})$$

$$\frac{\vdash \Gamma, \underline{\Delta}, A}{\vdash \Gamma, \underline{\Delta}, \underline{A}}\ (\mathrm{D}) \qquad \frac{\vdash \Gamma, \underline{\Delta}}{\vdash \Gamma, \underline{\Delta}, \underline{A}}\ (\mathrm{W})$$

$$\frac{\vdash \Gamma, \underline{\Delta}, \underline{A}, \underline{A}}{\vdash \Gamma, \underline{\Delta}, \underline{A}}\ (\mathrm{C})$$

$$\frac{\vdash \Gamma, \underline{\Delta}, A \qquad \vdash \Gamma', \underline{\Delta'}, B}{\vdash \Gamma, \underline{\Delta}, \Gamma', \underline{\Delta'}, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma, \underline{\Delta}, A, B}{\vdash \Gamma, \underline{\Delta}, A \parr B}\ \parr$$

$$\frac{\vdash \underline{\Delta}, A \qquad \vdash \Gamma', \underline{\Delta'}, B}{\vdash \Gamma', \underline{\Delta'}, \Delta, A \olessthan B}\ \olessthan \qquad \frac{\vdash \Gamma, \underline{\Delta}, \underline{A}, B}{\vdash \Gamma, \underline{\Delta}, A \ltimes B}\ \ltimes$$

Figure 1: sequent calculus in Girard's transcendental syntax

We translate this sequent calculus into a subsystem of MELL. Before doing so, we briefly comment on the inference rules of the calculus.

Following Girard, underlining is a way to handle structural rules, and in this sense the underlined formula is treated as hidden, or invisible (ibid:842). Since we focus on translating the system into an existing logical framework, we do not employ the mechanism of underlining. Accordingly, an underlined formula $\underline{A}$ (resp. an underlined context $\underline{\Delta}$) is translated as $?A$ (resp. $?\Delta$).

With respect to the cut rules, Girard states that "the two cut rules are nothing but some claim about the conclusion, namely that the configuration $[A \otimes \sim A]$ or $\underline{[A \otimes \sim A]}$ can be eliminated" (ibid:848). Instead of these bracket-based cut rules, we adopt the standard cut rule.

Concerning the rules for $\olessthan$ and $\ltimes$, as indicated by the definitions $A \olessthan B := \neg(A \Rightarrow \neg B)$ (i.e. $!A \otimes B$) and $A \ltimes B := \neg A \Rightarrow B$ (i.e. $?A \parr B$) in (ibid:841), these rules amount to nothing more than restrictions on the application of the $\otimes$ and $\parr$ rules. Accordingly, we treat $\olessthan$ and $\ltimes$ merely as aliases arising from restricted uses of $\otimes$ and $\parr$, and we therefore do not include inference rules for $\olessthan$ and $\ltimes$ in our system. Moreover, the presence of two distinct rules for $\olessthan$ and $\ltimes$ gives rise to an additional issue, namely that it is unclear whether the $\ltimes$ rule is applicable to formulas containing modalities. In particular, Girard (2015) does not address whether the $\otimes$ or $\parr$ rules themselves are applicable to formulas involving modalities.

A key feature of the sequent calculus in Girard's transcendental syntax is the absence of any rule corresponding to the promotion rule. Instead, in addition to $\otimes$, the calculus includes the rule for $\olessthan$.

From the discussion above, the sequent calculus in Girard's transcendental syntax can be translated into the subsystem of MELL obtained by omitting the promotion rule.

In this paper, we denote this subsystem by MELL$^-$. The inference rules of MELL$^-$ are given in Figure 2.

$$\frac{}{\vdash A, \sim A}\ \text{Ax} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, \sim A}{\vdash \Gamma, \Delta}\ \textsc{Cut}$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ \text{D}$$

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?C \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?W$$

Figure 2: Sequent calculus rules for one-sided $\mathsf{MELL}^-$

In Girard (2015:832), it is stated that "our new exponentials $A \oslash B :=!A \otimes B$ and $A \ltimes B := ?A \parr B$ are De Morgan variants of intuitionistic implication: $A \oslash B := \neg(A \Rightarrow \neg B)$ and $A \ltimes B := \neg A \Rightarrow B$". However, interpreting $\oslash$ and $\ltimes$ directly as $A \oslash B :=!A \otimes B$ and $A \ltimes B :=?A \parr B$ raises the following difficulty. Indeed, one would expect $\neg(A \ltimes B) :=!\neg A \parr \neg B$, but since no promotion rule is available, the duality between $\oslash$ and $\ltimes$ cannot be properly defined. The duality between $A \oslash B$ and $A \ltimes B$ is expected to be reflected in the duality of their subformulas $A$ and $B$. However, in the absence of the promotion rule, this subformula-level duality fails to hold.

Moreover, the proof-theoretic strength of $\mathsf{MELL}^-$ is severely limited. These failures are not accidental: they can be traced back to the absence of the promotion rule, which is responsible for lifting linear entailments to modal entailments in standard linear logic. Although transcendental syntax is presented as a retreat toward intuitionistic implication, the absence of promotion prevents $\oslash$ and $\ltimes$ from functioning as genuine internal homs. In particular, they fail to support the usual modal reasoning principles that underlie intuitionistic implication. Even when the connectives $\oslash$ and $\ltimes$ are introduced, the system fails to validate several basic properties that are standardly associated with modalities in linear logic.

In particular, the following sequents are not derivable in $\mathsf{MELL}^-$:

1. Functoriality of !:

$$\vdash !(A \multimap B) \otimes !A \multimap !B \qquad (\equiv \vdash\ ?(A \otimes \sim B) \parr \sim A,\ !B)$$

2. Internalization of implication:

$$!(A \multimap B) \vdash !A \multimap !B \qquad (\equiv \vdash\ ?(A \otimes \sim B),\ ?\sim A \parr !B)$$

3. Idempotence of !:

$$!A \vdash !!A \qquad (\equiv \vdash\ ?\sim A,\ !!A)$$

4. Comonoidal structure of !:

$$!A \vdash !A \otimes !A \qquad (\equiv \vdash\ ?\sim A,\ !A \otimes !A)$$

Consequently, the expressive power of modalities in $\mathsf{MELL}^-$ is strictly weaker than that of standard $\mathsf{MELL}$: the exponential ! fails to behave either as a functor or as a comonoidal modality.

From a proof-theoretic perspective, transcendental syntax should therefore be understood not as a conservative restriction of linear logic, but as a system with a fundamentally different modal behavior.

This observation does not diminish the conceptual originality of transcendental syntax, whose primary aim lies in the interaction between proof-nets and unification theory. Rather, it clarifies the precise proof-theoretic cost of the restrictions imposed on modalities.